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M.A./M.Sc. FIRST SEMESTER EXAMINATION, JANUARY 2015

FIRST YEAR

MATHEMATICS

Date : 14/01/2015

Time : 11 am – 1 pm

Paper : II

Full Marks : 50

[*The paper carries 58 marks and you can score maximum 50. You state the result or theorem clearly whichever you use.*]

1. Let F be a field of characteristic zero and let V be a finite dimensional vector space over F. If $\alpha_1, ..., \alpha_m$ are finitely many vectors in V, each different from the zero vector, prove that there is a linear functional f on V such that $f(\alpha_i) \neq 0 \; i = 1, 2, ..., m$. [7]

2. a) Let P be a prime. Find the order of the group $GL_n(\mathbb{Z}_P)$ (i.e, set of all n×n invertible matrices) [3]

   b) If M and N are subspaces of a finite-dimensional vector space, then prove that $(M \cap N)^0 = M^0 + N^0$ [5]

Or,

Suppose that A and B are two linear transformations on a finite dimensional vector space.

a) Prove that if A is a projection, then AB and BA have the same characteristic polynomial. [5]

[**Hint:** Choose a basis that makes the matrix of A as simple as possible and then compute directly with the matrices]

b) Prove that, in all cases, AB and BA have the same characteristic polynomial.

[**Hint:** Find an invertible R such that RA is a projection and apply (a) to RA & BR^{-1}] [3]

3. Prove that two commutative linear transformations on a finite-dimensional vector space V over an algebrically closed field can be simultaneously triangularized. [8]

Or,

a) Suppose that V is a finite-dimensional vector space over a field F of characteristic zero. If f and g are two linear functionals on V such that g(x) = 0 whenever f(x) = 0, then prove that f and g are linearly dependent. [4]

b) If W is a k-dimensional subspace of an n-dimensional vector space, then prove that W is the intersection of (n – k) hyperspace in V. [4]

4. Suppose that T is a linear transformation on a finite-demensional vectorspace V over a field of characteristic zero, and suppose that trace (T) = 0 w.r.t some basis β of V. Then prove that $[T]_\beta$ (matrix of T) is similar to a matrix A whose diagonal entries are zero. [8]

Or,

Find the Jordan Form of $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$. [8]

5. If F is field of characteristic zero, and T is a linear transformation on a finite-dimensional vector space V over F such that trace $(T^K) = 0$ for all $K \geq 1$, then prove that T is nilpotent. [7]

Or,

State and prove the Cauchy-Schwarz's inequality for an inner product space. (You are not allowed to assume any result) [7]

6. a) If x and y are two vectors in an Euclidean space, then prove that $\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$.

   Write the geometric interpretation of this result. [4]

   b) Suppose that $T_1, .., T_K$ be K linear transformations on a finite dimensional inner product space. If $\sum_{\ell=1}^{K} \text{trace}(T_\ell^* T_\ell) = 0$, then prove that $T_1 = T_2 = ... = T_K = 0$. [4]

Or,

   a) If a unitary matrix is triangular, then prove that it is diagonal. [5]

   b) If T is a skew-linear transformation on an inner product space, then prove that $(T - I)$ is invertible. [3]

7. Prove that two perpendicular projections P & Q on an inner product space are orthogonal if and only if the range of P and range of Q are perpendicular. [7]

Or,

State precisely the spectral theorem for normal linear transformations. Use spectral decomposition of a normal transformation on a complex inner product space to prove that if M is an invariant subspace (of a complex inner product space) under T, then $M^\perp$ is also invariant under T. [7]

8. If T is normal and if $T^3 = T^2$, then prove that T is idempotent. Does the conclusion remain true if the assumption of normality is omitted? [5]

Or,

If T is self-adjoint and $T^K = 1$ for some strictly positive integer K, then prove that T is involutory. [5]

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